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ON THE SOME IDENTITIES OF THE TYPE 2 DAEHEE AND CHANGHEE POLYNOMIALS ARISING FROM p-ADIC INTEGRALS ON $\mathbb{Z}_p$

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Abstract. Recently, Kim et al. introduced the type 2 Daehee and Changhee polynomials and give some new identities for these polynomials and numbers. In this paper, we study type 2 Daehee and Changhee polynomials and numbers arising from p-adic integrals on $\mathbb{Z}_p$. Regarding to those polynomials and numbers, we investigate some identities, Witt's formula, and distribution relation of these polynomials and numbers.

1. Introduction

Let p be a fixed prime number with $p \equiv 1(mod\, 2)$. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively.

Let $f(x)$ be uniformly differential function on $\mathbb{Z}_p$. Then the bosonic p-adic integral on $\mathbb{Z}_p$ is defined by Kim to be

$$\int_{\mathbb{Z}_p} f(x)d\mu(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad \text{(see [3, 4, 6, 8, 10, 11])}. \tag{1.1}$$

Thus, by (1.1), we get

$$\int_{\mathbb{Z}_p} f(x+1)d\mu(x) = \int_{\mathbb{Z}_p} f(x)d\mu(x) + f'(0). \tag{1.2}$$

The Daehee polynomials are defined by the generating function to be

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}, \quad \text{(see [4, 6, 8, 9, 10, 11, 17])}. \tag{1.3}$$

When $x = 0$, $D_n = D_n(0)$ are called the Daehee numbers.

In [9], Kim et al. introduced the type 2 Daehee polynomials are defined by the generating function to be

$$\frac{\log(1+t)}{(1+t)-(1+t)^{-1}}(1+t)^x = \sum_{n=0}^{\infty} d_n(x)\frac{t^n}{n!}. \tag{1.4}$$

When $x = 0$, $d_n = d_n(0)$ are called the type 2 Daehee numbers.

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It is well known that Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\ \ (\text{see } [1,2,5,12,19]). \tag{1.5}$$

When $x=0$, $B_n=B_n(0)$ are called the Bernoulli numbers.

In [5,7,9], Kim et al. introduced the type 2 Bernoulli polynomials which are given by the generating function to be

$$\frac{t}{2}\ csch\ t\ e^{xt}=\frac{t}{e^t-e^{-t}}e^{xt}=\sum_{n=0}^{\infty}b_n(x)\frac{t^n}{n!}. \tag{1.6}$$

When $x=0$, $b_n=b_n(0)$ are called the type 2 Bernoulli numbers.

Let $f(x)$ be continuous function on $\mathbb{Z}_p$. Then the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim to be

$$\int_{\mathbb{Z}_p}f(x)d\mu_{-1}(x)=\lim_{N\to\infty}\sum_{x=0}^{p^N-1}f(x)(-1)^x,\ \ (\text{see } [13,14,16,18,20]). \tag{1.7}$$

Thus, by (1.7), we get

$$\int_{\mathbb{Z}_p}f(x+1)d\mu_{-1}(x)+\int_{\mathbb{Z}_p}f(x)d\mu_{-1}(x)=2f(0). \tag{1.8}$$

The Changhee polynomials are defined by the generating function to be

$$\frac{2}{2+t}(1+t)^x=\sum_{n=0}^{\infty}Ch_n(x)\frac{t^n}{n!},\ \ (\text{see } [9,15,16,18,20]). \tag{1.9}$$

When $x=0$, $Ch_n=Ch_n(0)$ are called the Changhee numbers.

In [9], Kim et al. introduced type 2 Changhee polynomials are defined by the generating function to be

$$\frac{2}{(1+t)+(1+t)^{-1}}(1+t)^x=\sum_{n=0}^{\infty}C_n(x)\frac{t^n}{n!}. \tag{1.10}$$

When $x=0$, $C_n=C_n(0)$ are called the type 2 Changhee numbers.

It is well known Euler polynomials which are given by the generating function to be

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\ \ (\text{see } [1,2,5,13,14,19]). \tag{1.11}$$

When $x=0$, $E_n=E_n(0)$ are called the Euler numbers.

In [5,7,9], Kim et al. introduced the type 2 Euler polynomials which are given by the generating function to be

$$sech\ t\ e^{xt}=\frac{2}{e^t+e^{-t}}e^{xt}=\sum_{n=0}^{\infty}e_n(x)\frac{t^n}{n!}. \tag{1.12}$$

When $x=0$, $e_n=e_n(0)$ are called the type 2 Euler numbers.

The Stirling numbers of the first kind is defined by the generating function to be

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \quad \text{(see [5, 7, 9])}, \tag{1.13}$$

and the Stirling numbers of the second kind is defined by the generating function to be

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \quad \text{(see [5, 7, 15])}. \tag{1.14}$$

In [9], Kim et al. introduced the type 2 Changhee and Daehee polynomials and give some new identities for these polynomials and numbers. In this paper, we study type 2 Daehee and Changhee polynomials and numbers arising from p-adic integrals on $\mathbb{Z}_p$. Regarding to those polynomials and numbers, we investigate some identities, Witt's formula, and distribution relation of these polynomials and numbers.

2. Some identities of the type 2 Daehee and Changhee polynomials arising from p-adic integrals on $\mathbb{Z}_p$

In the following discussions, we assume that $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$. In the viewpoint of (1.2), we define the type 2 Daehee polynomials from p-adic integral on $\mathbb{Z}_p$ as follows;

$$\begin{aligned} \frac{1}{2}\int_{\mathbb{Z}_p}(1+t)^{2y+x+1}d\mu(y) &= \frac{\log(1+t)}{(1+t)-(1+t)^{-1}}(1+t)^x \\ &= \sum_{n=0}^{\infty} d_n(x)\frac{t^n}{n!}. \end{aligned} \tag{2.1}$$

When $x=0$, $d_n = d_n(0)$ are called the type 2 Daehee numbers.

On the other hand,

$$\begin{aligned} \frac{1}{2}\int_{\mathbb{Z}_p}(1+t)^{2y+x+1}d\mu(y) &= \frac{1}{2}\int_{\mathbb{Z}_p} e^{(2y+x+1)\log(1+t)}d\mu(y) \\ &= \frac{1}{2}\sum_{k=0}^{\infty}\int_{\mathbb{Z}_p}(2y+x+1)^k d\mu(y)\frac{1}{k!}\Big(\log(1+t)\Big)^k \\ &= \frac{1}{2}\sum_{k=0}^{\infty}\int_{\mathbb{Z}_p}(2y+x+1)^k d\mu(y)\sum_{n=k}^{\infty}S_1(n,k)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\frac{1}{2}\int_{\mathbb{Z}_p}(2y+x+1)^k d\mu(y)S_1(n,k)\Big)\frac{t^n}{n!}. \end{aligned} \tag{2.2}$$

Where $S_1(n,k)$ is the Stirling number of the first kind.

From (2.1) and (2.2), we have the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$d_n(x) = \sum_{k=0}^{n} \frac{1}{2} \int_{\mathbb{Z}_p} (2y + x + 1)^k d\mu(y) S_1(n, k). \tag{2.3}$$

Since

$$\begin{aligned} \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!} &= \frac{t}{e^t - e^{-t}} e^{xt} \\ &= \frac{1}{2} \int_{\mathbb{Z}_p} e^{(2y+x+1)t} d\mu(y) \\ &= \sum_{n=0}^{\infty} \frac{1}{2} \int_{\mathbb{Z}_p} (2y + x + 1)^n d\mu(y) \frac{t^n}{n!} \end{aligned} \tag{2.4}$$

From Theorem 2.1 and (2.4), we have the following Corollary.

Corollary 2.2. *For $n \geq 0$, we have*

$$d_n(x) = \sum_{k=0}^{n} b_k(x) S_1(n, k). \tag{2.5}$$

Now, we observe that

$$\begin{aligned} \int_{\mathbb{Z}_p} (1 + t)^{2y+x+1} d\mu(y) &= 2 \frac{\log(1 + t)}{(1 + t) - (1 + t)^{-1}} (1 + t)^x \\ &= 2 \sum_{n=0}^{\infty} d_n(x) \frac{t^n}{n!}. \end{aligned} \tag{2.6}$$

On the other hand,

$$\begin{aligned} \frac{\log(1 + t)}{(1 + t) - (1 + t)^{-1}} (1 + t)^x &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \binom{2y + x + 1}{n} d\mu(y) t^n \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (2y + x + 1)_n d\mu(y) \frac{t^n}{n!}, \end{aligned} \tag{2.7}$$

where $(x)_n = x(x - 1) \cdots (x - n + 1)$.

Therefore, by (2.6) and (2.7), we have the following theorem.

Theorem 2.3. ***(Witt's formula for** $d_n(x)$**)***
For $n \geq 0$, we have

$$2\ d_n(x) = \int_{\mathbb{Z}_p} (2y + x + 1)_n d\mu(y). \tag{2.8}$$

In particular,

$$2\ d_n = \int_{\mathbb{Z}_p} (2x + 1)_n d\mu(x). \tag{2.9}$$

By replacing t by e^t-1 in (2.1), we get

$$\begin{aligned}\sum_{k=0}^{\infty} d_k(x)\frac{1}{k!}(e^t-1)^k &= \frac{1}{2}\int_{\mathbb{Z}_p} e^{(2y+x+1)t}d\mu(y)\\ &= \frac{t}{e^t-e^{-t}}e^{xt}\\ &= \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}.\end{aligned} \tag{2.10}$$

On the other hand,

$$\begin{aligned}\sum_{k=0}^{\infty} d_k(x)\frac{1}{k!}(e^t-1)^k &= \sum_{k=0}^{\infty} d_k(x)\sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} d_k(x)S_2(n,k)\Big)\frac{t^n}{n!}.\end{aligned} \tag{2.11}$$

Therefore, by (2.10) and (2.11), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$b_n(x) = \sum_{k=0}^{n} d_k(x)S_2(n,k). \tag{2.12}$$

We observe that

$$\begin{aligned}\int_{\mathbb{Z}_p} f(x)d\mu(x) &= \lim_{N\to\infty}\frac{1}{p^N}\sum_{x=0}^{p^N-1} f(x)\\ &= \lim_{N\to\infty}\frac{1}{dp^N}\sum_{x=0}^{dp^N-1} f(x)\\ &= \frac{1}{d}\lim_{N\to\infty}\frac{1}{p^N}\sum_{a=0}^{d-1}\sum_{x=0}^{p^N-1} f(a+dx)\\ &= \frac{1}{d}\sum_{a=0}^{d-1}\int_{\mathbb{Z}_p} f(a+dx)d\mu(x).\end{aligned} \tag{2.13}$$

Proposition 1. *For $d \in \mathbb{N}$, we have*

$$\int_{\mathbb{Z}_p} f(x)d\mu(x) = \frac{1}{d}\sum_{a=0}^{d-1}\int_{\mathbb{Z}_p} f(a+dx)d\mu(x). \tag{2.14}$$

By (2.14), we note that

$$\begin{aligned}\sum_{n=0}^{\infty} d_n(x)\frac{t^n}{n!} &= \frac{1}{2}\int_{\mathbb{Z}_p}(1+t)^{2y+x+1}d\mu(y)\\ &= \frac{1}{2}\frac{1}{d}\sum_{a=0}^{d-1}\int_{\mathbb{Z}_p}(1+t)^{2(a+dy)+x+1}d\mu(y)\\ &= \frac{1}{2}\frac{1}{d}\sum_{a=0}^{d-1}(1+t)^{2a+x+1}\int_{\mathbb{Z}_p}(1+t)^{2dy}d\mu(y)\\ &= \frac{1}{d}\sum_{a=0}^{d-1}\frac{d\log(1+t)}{e^{d\log(1+t)}-e^{-d\log(1+t)}}e^{(\frac{2a+x+1}{d}-1)d\log(1+t)}\end{aligned} \tag{2.15}$$

$$\begin{aligned}&= \frac{1}{d}\sum_{a=0}^{d-1}\sum_{k=0}^{\infty} b_k(\frac{2a+x+1}{d}-1)\frac{1}{k!}d^k\Big(\log(1+t)\Big)^k\\ &= \frac{1}{d}\sum_{a=0}^{d-1}\sum_{k=0}^{\infty} b_k(\frac{2a+x+1}{d}-1)\sum_{n=k}^{\infty}S_1(n,k)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}d^{k-1}\sum_{a=0}^{d-1}b_k(\frac{2a+x+1}{d}-1)S_1(n,k)\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore, by (2.15), we get the following theorem.

Theorem 2.5. *For $n \geq 0$ and $d \in \mathbb{N}$, we have*

$$d_n(x) = \sum_{k=0}^{n}d^{k-1}\sum_{a=0}^{d-1}b_k(\frac{2a+x+1}{d}-1)S_1(n,k). \tag{2.16}$$

In the viewpoint of (1.8), we define the type 2 Changhee polynomials from p-adic integrals on $\mathbb{Z}_p$ as follows;

$$\begin{aligned}\int_{\mathbb{Z}_p}(1+t)^{2y+x+1}d\mu_{-1}(y) &= (1+t)^{x+1}\int_{\mathbb{Z}_p}(1+t)^{2y}d\mu_{-1}(y)\\ &= (1+t)^{x+1}\frac{2}{(1+t)^2+1}\\ &= \frac{2}{(1+t)+(1+t)^{-1}}(1+t)^x\\ &= \sum_{n=0}^{\infty}C_n(x)\frac{t^n}{n!}.\end{aligned} \tag{2.17}$$

When $x=0$, $C_n = C_n(0)$ are called the type 2 Changhee numbers.

Note that

$$
\begin{aligned}
\int_{\mathbb{Z}_p}(1+t)^{2y+x+1}d\mu_{-1}(y) &= \int_{\mathbb{Z}_p} e^{(2y+x+1)\log(1+t)}d\mu_{-1}(y) \\
&= \sum_{k=0}^{\infty}\int_{\mathbb{Z}_p}(2y+x+1)^k d\mu_{-1}(y)\frac{1}{k!}\Big(\log(1+t)\Big)^k \\
&= \sum_{k=0}^{\infty}\int_{\mathbb{Z}_p}(2y+x+1)^k d\mu_{-1}(y)\sum_{n=k}^{\infty}S_1(n,k)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\int_{\mathbb{Z}_p}(2y+x+1)^k d\mu_{-1}(y)S_1(n,k)\Big)\frac{t^n}{n!}.
\end{aligned} \tag{2.18}
$$

From (2.17) and (2.18), we have the following theorem.

Theorem 2.6. *For $n \geq 0$, we have*

$$
C_n(x) = \sum_{k=0}^{n}\int_{\mathbb{Z}_p}(2y+x+1)^k d\mu_{-1}(y)S_1(n,k). \tag{2.19}
$$

Observe that

$$
\begin{aligned}
\sum_{n=0}^{\infty}e_n(x)\frac{t^n}{n!} &= \frac{2}{e^t+e^{-t}}e^{xt} \\
&= \int_{\mathbb{Z}_p} e^{(2y+x+1)t}d\mu_{-1}(y) \\
&= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}(2y+x+1)^n d\mu_{-1}(y)\frac{t^n}{n!}.
\end{aligned} \tag{2.20}
$$

Where $e_n(x)$ are called the type 2 Euler polynomials.

From Theorem 2.6 and (2.20), we have the following Corollary.

Corollary 2.7. *For $n \geq 0$, we have*

$$
C_n(x) = \sum_{k=0}^{n}e_k(x)S_1(n,k). \tag{2.21}
$$

By replacing t by $e^t - 1$ in (2.17), we get

$$
\begin{aligned}
\int_{\mathbb{Z}_p} e^{(2y+x+1)t}d\mu_{-1}(y) &= \frac{2}{e^t+e^{-t}}e^{xt} \\
&= \sum_{n=0}^{\infty}e_n(x)\frac{t^n}{n!}.
\end{aligned} \tag{2.22}
$$

On the other hand,

$$\begin{aligned}\sum_{k=0}^{\infty} C_k(x)\frac{1}{k!}(e^t-1)^k &= \sum_{k=0}^{\infty} C_k(x)\sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} C_k(x)S_2(n,k)\Big)\frac{t^n}{n!}.\end{aligned} \tag{2.23}$$

Therefore, by (2.22) and (2.23), we obtain the following theorem.

Theorem 2.8. *For $n \geq 0$, we have*

$$e_n(x) = \sum_{k=0}^{n} C_k(x)S_2(n,k). \tag{2.24}$$

Now, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty} d_n(x)\frac{t^n}{n!} &= \int_{\mathbb{Z}_p}(1+t)^{2y+x+1}d\mu_{-1}(y) \\ &= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\binom{2y+x+1}{n}d\mu_{-1}(y)t^n \\ &= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}(2y+x+1)_n d\mu_{-1}(y)\frac{t^n}{n!}.\end{aligned} \tag{2.25}$$

Therefore, by (2.25), we have the following theorem.

Theorem 2.9. ***(Witt's formula for** $C_n(x)$**)***
For $n \geq 0$, we have

$$C_n(x) = \int_{\mathbb{Z}_p}(2y+x+1)_n d\mu_{-1}(y). \tag{2.26}$$

In particular,

$$C_n = \int_{\mathbb{Z}_p}(2x+1)_n d\mu_{-1}(y). \tag{2.27}$$

For $d \in \mathbb{N}$ with $d \equiv 1 (mod\ 2)$ we have

$$\begin{aligned} \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x \\ &= \lim_{N\to\infty} \sum_{x=0}^{dp^N-1} f(x)(-1)^x \\ &= \sum_{a=0}^{d-1} \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(a+dx)(-1)^{a+dx} \\ &= \sum_{a=0}^{d-1} (-1)^a \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(a+dx)(-1)^x \\ &= \sum_{a=0}^{d-1} (-1)^a \int_{\mathbb{Z}_p} f(a+dx) d\mu_{-1}(x). \end{aligned} \tag{2.28}$$

Proposition 2. *For $d \in \mathbb{N}$ with $d \equiv 1 (mod\ 2)$ we have*

$$\int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \sum_{a=0}^{d-1} (-1)^a \int_{\mathbb{Z}_p} f(a+dx) d\mu_{-1}(x). \tag{2.29}$$

By (2.29), we have

$$\begin{aligned} \sum_{n=0}^{\infty} C_n(x) \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} (1+t)^{2y+x+1} d\mu_{-1}(y) \\ &= \sum_{a=0}^{d-1} (-1)^a \int_{\mathbb{Z}_p} (1+t)^{2(a+dy)+x+1} d\mu_{-1}(y) \\ &= \sum_{a=0}^{d-1} (-1)^a (1+t)^{2a+x+1} \int_{\mathbb{Z}_p} (1+t)^{2dy} d\mu_{-1}(y) \\ &= \sum_{a=0}^{d-1} (-1)^a (1+t)^{2a+x+1-d} \frac{2}{(1+t)^d - (1+t)^{-d}} \\ &= \sum_{a=0}^{d-1} (-1)^a \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} e_k\Big(\frac{2a+x+1}{d} - 1\Big) d^k S_1(n,k) \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} d^k \sum_{a=0}^{d-1} (-1)^a e_k\Big(\frac{2a+x+1}{d} - 1\Big) S_1(n,k) \right) \frac{t^n}{n!}. \end{aligned} \tag{2.30}$$

Therefore, by (2.30), we get the following theorem.

Theorem 2.10. *For $n \geq 0$ and $d \in \mathbb{N}$ with $d \equiv 1 (mod\ 2)$, we have*

$$C_n(x) = \sum_{k=0}^{n} d^k \sum_{a=0}^{d-1} (-1)^a e_k\Big(\frac{2a+x+1}{d} - 1\Big) S_1(n,k). \tag{2.31}$$

3. Conclusion

In recent years, Kim et al. introduced the various type 2 special polynomials and numbers and provided some identities and properties of those polynomials and numbers. In this paper, we study type 2 Daehee and Changhee polynomials arising from p-adic integrals on $\mathbb{Z}_p$. We represent Witt's formula type 2 Daehee and Changhee polynomials arising from p-adic invariant integral on $\mathbb{Z}_p$ in Theorem 2.3 and Theorem 2.9 respectively. Moreover, we investigate some explicit identities and properties related to type 2 Bernoulli polynomials and Euler polynomials. We provide type 2 Bernoulli polynomials and Euler polynomials associated with type 2 Daehee and Changhee polynomials as the inversion form in Theorem 2.4 and Theorem 2.8. Also, we represent the distribution of type 2 Daehee and Changhee polynomials using Proposition 1 and 2.

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